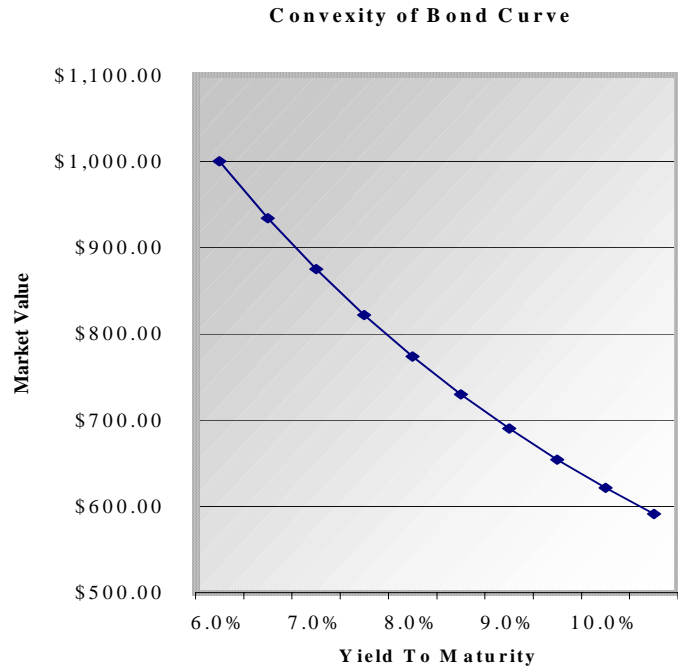


Bond Duration

We already know that the (Present) Value of any investment is the sum total of all future financial benefits, each discounted at a rate commensurate with the perceived risk. We know, too, that as the receipt of a cash benefit is pushed farther and farther into the future, the present value of that benefit diminishes. Bonds are no exception.

Therefore the risk of recovering the full reversionary value of the bond increases with the time to maturity. But in the interim, a percent of the purchase price may be recovered as a function of the coupon rate of the bond. A bond with a higher coupon rate – say 10%, or \$100 per year - will return a higher percentage of the bond’s current market value over a given number of years when compared to a bond of similar maturity but with a lower coupon rate, say 7% per year.



The calculation of **Bond Duration** brings all these factors together in one number, allowing us to have a measurement of a bond’s price sensitivity to changes in market interest rates.

Derivation of Macaulay's Duration Factor

We can represent the present value, or current market price, of the bond as:

$$\text{Value} = \sum_{t=1}^n \frac{CF_t}{(1+i)^t}$$

where

- CF** = coupon payment per period **t**
- i** = current market yield rate per period of time (annual rate/2)
- t** = time (expressed in 6-month periods)
- n** = number of 6-month periods to maturity

If we seek to measure the sensitivity of the Value (V) of the bond in response to changes in market yield rates (i), we need only take the first differentiation of V with respect to i :

$$\frac{dV}{di} = \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^{t+1}}$$

Note that Duration always carries a negative sign because of the sign of t. Since t is expressed in years, Duration is also expressed in years. But Duration is not a payback period, nor does it represent time to maturity.

When the value of i is small, as it will be when changes in market yield rates are small, the expression $\frac{1}{(1+i)}$ substantially equates to 1. To simplify the matter then, let's move one $\frac{1}{(1+i)}$ factor outside the summation portion of the formula, leaving:

$$\frac{dV}{di} = \frac{1}{(1+i)} \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t}$$

Then, since it substantially equates to 1, we can temporarily ignore it:

$$\frac{dV}{di} = 1 * \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t} = \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t}$$

But the expression $\frac{CF_t}{(1+i)^t}$ represents the Present Value of the particular cashflow CF_t.

Therefore we can restate the equation as approximately $\frac{dV}{di} = \sum_{t=1}^n -t * (PV)CF_t$

In 1938, Frederick R. Macaulay defined Duration as the *total weighted average time for recovery of the payments and principal in relation to the current market price of the bond.* Bond Duration, therefore, is

$$\text{Duration} = \sum_{t=1}^n \frac{-t * (PV)CF_t}{\text{Market Price}}$$

$$\text{where Market Price} = \sum_{t=1}^n \frac{CF_t}{(1+i)^t}$$

and (PV)CF_t = the Present Value of cashflow t.

How Bond Duration is Calculated

The calculation of a bond's Duration was a time-consuming task in Macaulay's day. Today the computer makes the measurement of bond value as a result in a change in market yield - or any other variable - a relatively minor chore. Yet, bond Duration is still a valuable strategic tool in the hands of the bond investor, especially in assembling a portfolio of bonds.

Following Frederick Macaulay's formula, Bond Duration for a 3-year bond, bearing a 6% coupon and a market yield of 10%, is calculated as:

	A	B	C	D	E	F	G
1	Year ¹	Pmt #	Coupon \$	PV Factor	\$PV	PV/Price	Duration ²
2	-0.50	1.00	\$30.00	0.952381	\$28.57	0.0318	-0.0159
3	-1.00	2.00	\$30.00	0.907029	\$27.21	0.0303	-0.0303
4	-1.50	3.00	\$30.00	0.863838	\$25.92	0.0288	-0.0433
5	-2.00	4.00	\$30.00	0.822702	\$24.68	0.0275	-0.0549
6	-2.50	5.00	\$30.00	0.783526	\$23.51	0.0262	-0.0654
7	-3.00	6.00	\$1030.00	0.746215	\$768.60	0.8554	-2.5663
8				Market Value =	\$898.49 ³	1.0000	-2.7761

Therefore this bond, with a current value of \$898.48, has a Duration of **-2.7761**.

The steps in calculating the Duration as it appears above are:

1. Determine the coupon rate. The coupon rate/2 * \$1000 = PMT (Coupon \$).
2. Determine the PV factor using the yield *per period*: $1/(1+i)^t$ where **t** is the PMT # and **i** is the annual interest rate/2
3. Multiply the **PV Factor** (d2) * **Coupon\$** (c2) to get the **\$PV** (E2) of the Coupon payment.
4. Add the **\$PVs** of the cashflows in column (E) to determine **Market Value** of the bond (E8)
5. Divide each result of step #3 (**\$PV**) by the current market value (E8) of the bond.
6. Multiply this factor (F2) by the **years** (A2) in column 1.

The **sum** of all final values in the right-hand column is the **Duration**.

Remember that Duration always carries a negative sign.

¹ The time in years is negative to conform to Macaulay's formula.

² Bond Duration is the product of **PV/Price** * the value under column **Year**. This is the reason that Duration is expressed in terms of years, but this is obviously not the capital pay-back period.

³ The Market Price is the summation of all the separate PVs in the cashflow.

Determinants of Duration

As we can see from the equations above, coupon rate (which determines the size of the periodic cashflow), yield (which determines present value of the periodic cashflow), and time-to-maturity (which weights each cashflow) all contribute to the Duration factor.

Holding coupon rate and maturity constant –

Increases in market yield rates cause a decrease in the present value factors of each cashflow. Since Duration is a product of the present value of each cashflow and time, higher yield rates also lower Duration. *Therefore Duration varies inversely with yield rates.*

Holding yield rate and maturity constant –

Increases in coupon rates raise the present value of each periodic cashflow and therefore the market price. This higher market price lowers Duration. *Therefore Duration varies inversely to coupon rate.*

Holding yield rate and coupon rate constant –

An increase in maturity increases Duration and cause the bond to be more sensitive to changes in market yields. Decreases in maturity decrease Duration and render the bond less sensitive to changes in market yield. *Therefore Duration varies directly with time-to-maturity (t).*

Using Duration and Modified Duration

The magnitude of the Duration is an index to the sensitivity of the bond to changes in market interest rates. Bonds with high Duration factors experience greater increases in value when rates decline, and greater losses in value when rates increase, compared to bonds with lower Duration.

In order to more closely *approximate* the percent change in market value as the result of a percent change in yield, Macaulay derived **Modified Duration**, which is simply Duration times the factor which we removed when we derived the formula for Duration above.

$$\text{Modified Duration (D}_M\text{)} = \text{Duration} * \frac{1}{(1+i)}$$

In the example above, where Duration is -2.7761 , the Modified Duration is:

$$\text{MDuration (D}_M\text{)} = -2.7761 * \frac{1}{(1 + \frac{0.10}{2})} = -2.6439$$

Note that the value of **i** (0.10) is the **annual** yield rate which must be divided by 2.

Macaulay used this Modified Duration, D_M , to *approximate* the percent change in bond value for a given percent change in yield, using the following formula:

$\text{Percent change in bond value} = D_M * \text{change in yield.}$

If yield rates rose from 10% to 10.5%, a 0.5% increase in rates, Macaulay's formula would predict a percent change in value as:

$$\begin{aligned} \text{Percent change in bond value} &= D_M * \text{numerical change in stated yield.}^4 \\ &= -2.6439 * (+0.5) \\ &= -1.3220\% \end{aligned}$$

The price change calculated by MDuration would be $\$898.49 * -1.322\% = -\11.88 . The new bond price would be approximately $\$898.49 - \$11.88 = \$886.61$. We can confirm the percent change and new price by entering these data into a spreadsheet: The change takes place in the **PV Factor** as a result of the change in market yield.

Year	Pmt #	Coupon	PV Factor	\$PV	\$PV/ Price	Duration
-0.50	1	\$30.00	0.95012	\$28.5036	0.03215	-0.01608
-1.00	2	\$30.00	0.90273	\$27.0818	0.03054	0.03054
-1.50	3	\$30.00	0.85770	\$25.7309	0.02902	-0.04353
-2.00	4	\$30.00	0.81491	\$24.4474	0.02757	-0.05514
-2.50	5	\$30.00	0.77426	\$23.2279	0.02620	-0.06550
-3.00	6	\$1030.00	0.73564	\$757.7128	0.85453	-2.5636
			Price	\$886.70	1.00000	-2.77438

As you can see, the computer indicates a decline in value from \$898.49 to \$886.70, a loss of \$11.79 vs. \$11.88 as predicted by Macaulay's approximation using Modified Duration.

This difference in the answer we have obtained is caused by the **convexity** of the bond value curve. Macaulay's formula describes a straight line, but bond value in response to yield changes describes a convex curve. When yield changes are small (as in this example), the difference in value change is negligible, but when these differences are substantial (larger percent changes in market yield and higher Duration) then the differences in value increase.

If the Duration of our example bond were in the order -8 or -12, an increase of 1.0% in interest rates would indicate a loss of approximately 8% (\$71.88) and 12% (\$107.82), respectively in bond price. But because of these large changes in yield, and the high Duration, the linearity of the Duration curve would result in larger pricing errors. Therefore the use of Duration to estimate change results is a reasonable *approximation*, especially when the changes in interest rates are not too large.

Significance and Use of Duration

In the pre-computer days of Macaulay, Duration was conceived as a short-hand method of estimating price volatility as the result of changes in market yield. Today, the value of Duration is somewhat less evident, since computer pricing programs are widely available which can indicate

⁴ Since Modified Duration is a negative value, a decrease in yield rate results in an increase in bond value. Multiplying the negative Duration times a decrease in yield results in an increase in bond value.

precisely the value of a bond with respect to all the important financial variables: coupon, yield and time. Still, Duration can be used by the bond investor to implement his investment strategy.

If the investor believes that market yields are going to decline, he may wish to alter his bond mix to include bonds carrying higher Durations in order to leverage the increase in bond value. If an increase in yields is expected, he may elect to change the mix to include bonds of lower Duration to minimize the negative effect on his portfolio..

Obviously bonds are subject to risk beyond changes in the coupon-yield-maturity variables, e.g. the risk of default, but Duration is not intended to reflect risk; it measures interest rate *sensitivity*.

Duration and the Bond Portfolio

Perhaps the most prevalent use of Duration today is as a short-hand method of estimating the potential changes in the value of a portfolio of bonds.

Assume that a portfolio consists - for simplicity's sake - of three bonds carrying the following current prices and Modified Durations:

<u>Bond</u>	<u>Current Price</u>	<u>Mod. Duration</u>
A	\$845.57	4.12257
B	\$625.95	7.3523
C	\$884.17	4.04855

On a given day the market yield increases 20 basis points (+ 0.2%). What effect will this have on the value of this portfolio? Fortunately, the HP-12C has a set of statistical registers which will calculate a weighted mean. Here are the keystrokes (set decimal to \boxed{f} 5):

<u>Key In</u>	<u>Display Shows</u>	<u>Comments</u>
-4.12257 $\boxed{\text{Enter}}$	-4.12257	Enters Mod. Duration
0.2 $\boxed{\%}$	-0.00825	Mod. Duration x % change
845.57 $\boxed{\Sigma+}$ (4,9)	1.00000	Puts price into statistical register
-7.3523 $\boxed{\text{Enter}}$	-7.3523	(same as above)
0.2 $\boxed{\%}$	-0.01470	
\$625.95 $\boxed{\Sigma+}$	2.00000	
-4.04855 $\boxed{\text{Enter}}$	-4.04855	
0.2 $\boxed{\%}$	-0.00810	
\$884.17 $\boxed{\Sigma+}$	3.00000	

Now, by recalling R₂ (3,8), you will retrieve the total of all the original bond prices:

RCL	2	2,355.69	Total of original prices.
RCL	6	-23.3354	The loss in value. (This is a loss since the rate increased).
	+	2,332.3546	Adds the loss to show the new value of the portfolio.
f	2	\$2,332.35	Re-sets price to 2 decimal places.

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