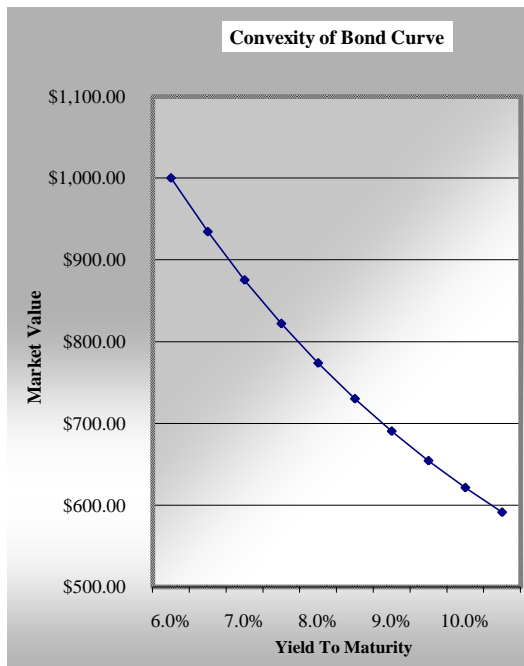


Bond Duration

We already know that the (Present) Value of any investment is the sum total of all future financial benefits, each discounted at a rate commensurate with the perceived risk. We know, too, that as the receipt of a cash benefit is pushed farther and farther into the future, the present value of that benefit diminishes. Bonds are no exception.

Therefore the risk of recovering the full reversionary value of the bond increases with the time to maturity. But in the interim, a percent of the purchase price may be recovered as a function of the coupon rate of the bond. A bond with a higher coupon rate – say 10%, or \$100 per year – will return a higher percentage of the bond’s current market value over a given number of years when compared to a bond of similar maturity but with a lower coupon rate, say 7% per year.



The calculation of **Bond Duration** brings all these factors together in one number, allowing us to have a measurement of a bond’s price sensitivity to changes in market interest rates.

Derivation of Macaulay's Duration Factor

We can represent the present value, or current market price, of the bond as:

$$\text{Value} = \sum_{t=1}^n \frac{CF_t}{(1+i)^t}$$

where **CF** = cashflow payment¹ per period **t**
i = current market yield rate per period of time (annual rate/2)
t = time (expressed in 6-month periods)
n = number of 6-month periods to maturity

If we seek to measure the sensitivity of the Value (*v*) of the bond in response to changes in market yield rates (**i**), we need only take the first differentiation of **V** with respect to **i** :

$$\frac{dv}{di} = \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^{t+1}}$$

Note that Duration always carries a negative sign because of the sign of **t**. Since **t** is expressed in years, Duration is also expressed in years. But Duration is not a payback period, nor does it represent time to maturity.

¹ The last cashflow, *n*, will include the maturity value of the bond

When the value of i is small, as it will be when changes in market yield rates are small, the expression $\frac{1}{(1+i)}$ *substantially* equates to 1. To simplify the matter then, let's move one $\frac{1}{(1+i)}$ factor outside the summation portion of the formula, leaving:

$$\frac{dv}{di} = \frac{1}{(1+i)} \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t}$$

Then, since it substantially equates to 1, we can temporarily ignore it:

$$\frac{dv}{di} = 1 * \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t} = \sum_{t=1}^n \frac{-t * CF_t}{(1+i)^t}$$

But the expression $\frac{CF_t}{(1+i)^t}$ represents the Present Value of the particular cashflow CF_t .

Therefore we can restate the equation as approximately $\frac{dv}{di} = \sum_{t=1}^n -t * (PV)CF_t$

In 1938, Frederick R. Macaulay defined Duration as the *total weighted average time for recovery of the payments and principal in relation to the current market price of the bond.*

Bond Duration, therefore, is

$$\text{Duration} = \sum_{t=1}^n \frac{-t * (PV)CF_t}{\text{Market Price}}$$

$$\text{where Market Price} = \sum_{t=1}^n \frac{CF_t}{(1+i)^t}$$

and $(PV)CF_t =$ the Present Value of cashflow t .

How Bond Duration is Calculated

The calculation of a bond's Duration was a time-consuming task in Macaulay's day. Today the computer makes the measurement of bond value as a result of a change in market yield - or any other variable - a relatively minor chore. Yet, bond Duration is still a valuable strategic tool in the hands of the bond manager, especially in assembling a portfolio of bonds.

Following Frederick Macaulay's formula, Bond Duration for a 3-year bond, bearing a 6% coupon and a market yield of 10%, is calculated as:

	A	B	C	D	E	F	G
1	Year²	Pmt #	Coupon \$	PV Factor	\$PV	PV/Price	Duration³
2	-0.50	1.00	\$30.00	0.952381	\$28.57	0.0318	-0.0159
3	-1.00	2.00	\$30.00	0.907029	\$27.21	0.0303	-0.0303
4	-1.50	3.00	\$30.00	0.863838	\$25.92	0.0288	-0.0433
5	-2.00	4.00	\$30.00	0.822702	\$24.68	0.0275	-0.0549
6	-2.50	5.00	\$30.00	0.783526	\$23.51	0.0262	-0.0654
7	-3.00	6.00	\$1030.00	0.746215	\$768.60	0.8554	-2.5663
8				Market Value =	\$898.49⁴	1.0000	-2.7761

Therefore this bond, with a current value of \$898.48, has a Duration of **-2.7761**.

The steps in calculating the Duration as it appears above are:

1. Determine the coupon rate. The coupon rate $\div 2 * \$100 = \text{PMT}$ (Coupon \$).
2. Determine the PV factor using the yield *per period*: $1/(1+i)^t$ where **t** is the PMT # and **i** is the annual interest rate $\div 2$.
3. Multiply the **PV Factor** (D2) * **Coupon\$** (C2) to get the **\$PV** (E2) of the Coupon payment.
4. Add the **\$PVs** of the cashflows in column (E) to determine **Market Value** of the bond (E8)
5. Divide each result of step #3 (**\$PV**) by the current market value (E8) of the bond.
6. Multiply this factor (F2) by the **years** (A2) in column 1.

The **sum** of all final values in the right-hand column is the **Duration**.

Remember that Duration always carries a negative sign.

Determinants of Duration

As we can see from the equations above, coupon rate (which determines the size of the periodic cashflow), yield (which determines present value of the periodic cashflow), and time-to-maturity (which weights each cashflow) all contribute to the Duration factor.

Holding coupon rate and maturity constant –

Increases in market yield rates cause a decrease in the present value factors of each cashflow. Since Duration is a product of the present value of each cashflow and time, higher yield rates also lower Duration. *Therefore Duration varies inversely with yield rates.*

Holding yield rate and maturity constant –

Increases in coupon rates raise the present value of each periodic cashflow and therefore the market price. This higher market price lowers Duration. *Therefore Duration varies inversely to coupon rate.*

Holding yield rate and coupon rate constant –

² The time in years is negative to conform to Macaulay's formula.

³ Bond Duration is the product of **PV/Price** times the value under column **Year**. This is the reason that Duration is expressed in terms of years, but this is obviously not the capital pay-back period.

⁴ The Market Price is the summation of all the separate PVs in the cashflow.

An increase in maturity increases Duration and causes the bond to be more sensitive to changes in market yields. Decreases in maturity decrease Duration and render the bond less sensitive to changes in market yield. Therefore *Duration varies directly with time-to-maturity (t)*.

Using Duration and Modified Duration

The magnitude of the Duration is an index to the sensitivity of the bond to changes in market interest rates. Bonds with high Duration factors experience greater increases in value when rates decline, and greater losses in value when rates increase, compared to bonds with lower Duration.

In order to more closely *approximate* the percent change in market value as the result of a percent change in yield, Macaulay derived **Modified Duration**, which is simply Duration times the factor which we removed when we derived the formula for Duration above.

$$\text{Modified Duration (D}_M) = \text{Duration} * \frac{1}{(1+i)}$$

In the example above, where Duration is -2.7761 , the Modified Duration is:

$$\text{MDuration (D}_M) = -2.7761 * \frac{1}{(1 + \frac{0.10}{2})} = -2.6439$$

Note that the value of **i** (0.10) is the **annual** yield rate which must be divided by 2.

Macaulay used this Modified Duration, D_M , to *approximate* the percent change in bond value for a given percent change in yield, using the following formula:

$$\text{Percent change in bond value} = -D_M * \text{change in yield.}$$

If yield rates rose from 10% to 10.5%, a 0.5% increase in rates, Macaulay's formula would predict a percent change in value as:

$$\begin{aligned} \text{Percent change in bond value} &= -D_M * \text{numerical change in stated yield.}^5 \\ &= -2.6439 * (+0.5) \\ &= -1.3220\% \end{aligned}$$

The price change calculated by MDuration would be $\$898.49 * -1.322\% = -\11.88 The new bond price would be approximately $\$898.49 - \$11.88 = \$886.61$. We can confirm the percent change and new price by entering these data into a spreadsheet: The change takes place in the **PV Factor**⁶ as a result of the change in market yield.

Year	Pmt #	Coupon	PV Factor	\$PV	\$PV/ Price	Duration
-0.50	1	\$30.00	0.95012	\$28.5036	0.03215	-0.01608
-1.00	2	\$30.00	0.90273	\$27.0818	0.03054	0.03054
-1.50	3	\$30.00	0.85770	\$25.7309	0.02902	-0.04353

⁵ Since Modified Duration is a negative value, a decrease in yield rate results in an increase in bond value. Multiplying the negative Duration times a decrease in yield results in an increase in bond value.

⁶ The **PV** factor is simply the PV of \$1.00, discounted at a specified rate over a defined number of periods.

-2.00	4	\$30.00	0.81491	\$24.4474	0.02757	-0.05514
-2.50	5	\$30.00	0.77426	\$23.2279	0.02620	-0.06550
-3.00	6	\$1030.00	0.73564	\$757.7128	0.85453	-2.5636
			Price	\$886.70	1.00000	-2.77438

As you can see, the computer indicates a decline in value from \$898.49 to \$886.70, a loss of \$11.79 vs. \$11.88 as predicted by Macaulay's approximation.

This difference in the answer we have obtained is caused by the **convexity** of the bond value curve. Macaulay's formula describes a straight line, but bond value in response to yield changes describes a convex curve. When yield changes are small (as in this example), the difference in value change is negligible, but when these differences are substantial (larger percent changes in market yield and higher Duration) then the differences in value increase.

If the Duration of our example bond were in the order -8 or -12, an increase of 1.0 % in interest rates would indicate a loss of approximately 8% (\$71.88) and 12% (\$107.82), respectively in bond price. But because of these large changes in yield, and the high Duration, the linearity of the Duration curve would result in larger pricing errors. Therefore the use of Duration to estimate change results is a reasonable *approximation*, especially when the changes in interest rates are not too large.

Significance and Use of Duration

In the pre-computer days of Macaulay, Duration was conceived as a short-hand method of estimating price volatility as the result of changes in market yield. Today, the value of Duration is somewhat less evident, since computer pricing programs are widely available which can indicate precisely the value of a bond with respect to all the important financial variables: coupon, yield and time. Still, Duration can be used by the bond investor to implement his investment strategy.

If the investor believes that market yields are going to decline, he may wish to alter his bond mix to include bonds carrying higher Durations in order to leverage the increase in bond value. If an increase in yields is expected, he may elect to change the mix to include bonds of lower Duration to minimize the negative effect on his portfolio.

Obviously bonds are subject to risk beyond changes in the coupon-yield-maturity variables, e.g. the risk of default, but Duration is not intended to reflect risk; it measures interest rate *sensitivity*.

Duration and the Bond Portfolio

Perhaps the most prevalent use of Duration today is as a short-hand method of estimating the potential changes in the value of a portfolio of bonds.

Assume that a portfolio consists - for simplicity's sake - of three bonds carrying the following current prices and Modified Durations:

<u>Bond</u>	<u>Current Price</u>	<u>Mod. Duration</u>
A	\$845.57	-4.12257
B	\$625.95	-7.3523
C	\$884.17	-4.04855

On a given day the market yield increases 20 basis points (+ 0.2%). What effect will this have on the value of this portfolio? Fortunately, the HP-12C has a set of statistical registers which will calculate a weighted mean. Here are the keystrokes (set decimal to \boxed{f} 5):

<u>Key In</u>	<u>Display Shows</u>	<u>Comments</u>
-4.12257 $\boxed{\text{Enter}}$	-4.12257	Enters Mod. Duration
0.2 $\boxed{\%}$	-0.00825	Mod. Duration x % change
845.57 $\boxed{\Sigma+}$ (4,9)	1.00000	Puts price into statistical register
-7.3523 $\boxed{\text{Enter}}$	-7.3523	Enters Mod. Duration
0.2 $\boxed{\%}$	-0.01470	Mod. Duration x % change
\$625.95 $\boxed{\Sigma+}$	2.00000	Puts price into statistical register
-4.04855 $\boxed{\text{Enter}}$	-4.04855	Enters Mod. Duration
0.2 $\boxed{\%}$	-0.00810	Mod. Duration x % change
\$884.17 $\boxed{\Sigma+}$	3.00000	Puts price into statistical register

Now, by recalling R₂ (3, 8), you will retrieve the total of all the original bond prices:

$\boxed{\text{RCL}}$ 2	2,355.69	Total of original prices.
$\boxed{\text{RCL}}$ 6	-23.3354	The loss in value. (This is a loss since the rate increased).
$\boxed{+}$	2,332.3546	Adds the loss to show the new value of the portfolio.
\boxed{f} 2 \$2,332.35		Sets new value to 2 decimal places.

Zero Coupon Bonds

“Stripped Bonds” refers to bonds which are “stripped” of either 1) their coupons (interest payments) or 2) their reversionary value.⁷ These strips can be bought to furnish a series of interest payments but no reversionary value, or to deliver one lump sum at the time of maturity but no intervening payments.

The most popular of the “strips” are standard U.S. Treasury bonds.⁸ Bonds are selected by security dealers, transferred to the Federal Reserve Bank in New York which creates the derivative instruments and returns the instruments to the bond dealer by *Fedwire*. One can buy both the stripped interest coupons and/or the stripped principal. The Wall Street Journal denotes the former as **ci**, while the stripped principals are earmarked **bp**. (Quotes are usually located at the end of the U.S. Treasury bond tables.) As *short-term* investments, Zero Coupon bonds (**bps**) are very volatile derivative instruments popular with those who prefer to bet on changes in long-term interest rates.

For example, assume that current long-term (29-year) rates are at 9.5%. The Present Value (Ask) of a stripped-coupon Treasury maturing in 29 years is quoted at 7:22, meaning 7.6875, or \$76.875 (per \$1000 of reversionary value).

A trader foresees long-term interest rates declining to the 7.5% range within three years. Therefore the receipt of \$1,000 in 26 years (29 – 3), discounted semi-annually at 7.5% will be:

n	i	PV	PMT	FV
26*2	7.5÷2	?	0	1,000
	Solving...	-\$147.44		

This will represent a semi-annual **yield** in three years of:

n	i	PV	PMT	FV
3*2	?	-76.875	0	147.44
Solving	11.465			

The annual yield will be $11.465 * 2 = 22.93\%$

A yield such as this is very tempting. But the door swings to and fro.

Suppose that yield rates *increase* 2% in three years. The results demonstrate the reward and the risk of the high leverage implicit in Zeros:

n	i	PV	PMT	FV
26*2	11.5 ÷ 2	?	0	1,000
	Solving..	-54.629		

⁷ The term STRIPS is an acronym for Separate Trading of Registered Interest and Principal of Securities, a Treasury program which allows bonds of maturities equal to 10 years or more to be transferred over Fedwire. This action has greatly reduced the cost of insurance customarily associated with transferring these derivatives.

⁸ Corporate bonds are also available as “strips.”

The yield now becomes:

n	i	PV	PMT	FV
3 *2	?	-76.875	0	54.629
Solving..	-5.53			

The annual yield will be $-5.53 * 2 = -11.06\%$

Duration Of Zero Coupon Bonds

This volatility of Zero Coupon bonds (stripped principal) can also be seen in their Duration. Since there are no coupon payments, the entire Duration calculation is a function of the last payment, which is the principal. In the case of a 30-year Zero Coupon bond, discounted over 30 years (60 periods) @10% (5% per period), the Duration is:

Year	Pmt #	Coupon	PV Factor	\$PV	\$PV ÷ Price	Duration
-0.5	1	0	0.95238	0	0	0
-1.0	2	0	0.90703	0	0	0s
↓			↓			↓
-30	60	1,000.00	0.05354	\$53.53	1.00	-30.00
			Total \$53.53			-30.00

For all practical purposes then, the Duration of a Zero Coupon bond is approximately equal to its maturity (in years). *The longer the maturity, the higher the Duration*, and the more sensitive the derivative is to even small changes in market yield. A high Duration factor indicates that long-term Zero Coupon bonds are *very* sensitive to changes in market rates. An absolute change of only $\pm 0.5\%$ in interest rates would result in approximately a $\pm 15\%$ variation in value for a 30-year Zero Coupon bond.⁹

Modified Duration of Zeros

The modified duration of a zero coupon bond is calculated in the same way as for other bonds:

$$D_M = \frac{\text{Duration}}{\left(1 + \frac{\text{annual yield}}{2}\right)} = \frac{-30}{\left(1 + \frac{.10}{2}\right)} = -28.57$$

Taxation of Zero Coupon Bonds

Although Zero Coupon bonds do not pay interest, they do result in *annual* taxable income because a Zero Coupon bond is treated as an Original Issue Discount bond (OID). The amount of the OID is the difference between the maturity value of the bond and the price at which the bond was acquired. The tax is levied on the annual increase in the value of the bond as it approaches maturity.

⁹ Zeros are always less than 30 years in maturity because of the time necessary to convert them from standard bonds.

The amount which is currently recognized as taxable is determined by a ratio: the ratio is determined by dividing the number of days for which the bond has been held (in the current tax year) by the number of days to maturity. This factor is applied to the excess of the maturity value of the bond over the acquisition cost. The result is that portion of the increased value of the bond which is currently subject to tax at the ordinary rate. (There is no long-term capital gains treatment available for these amounts. Gains are taxed as ordinary income)

For example, a bond acquired on July 1 for \$500 will mature in exactly 10 years. The maturity value is \$1,000. The ratio is 183 days/3650 days, or 0.05014, or 5.014%. The amount recognized is $(\$1,000 - \$500) \times 5.014\% = \$25.05$. This amount is reported as interest earned..

Because these instruments produce no income during the holding period, and because they do result in taxable income, Zero Coupon bonds are commonly held in tax-deferred accounts.

Zeros As an Investment & Planning Tool

If U.S. Treasury Zero Coupon bonds are acquired for long-term purposes, without the need for intermediate liquidity, they can be important low-risk instruments with which to fund future cashflow requirements. For example:

An assessment of future income requirements for an education fund indicates that \$250,000 will be required in 18 years. Zero Treasuries which will mature in 18 years are currently priced at 34:21, or \$346.5625, at a time when long term interest rates are 6.06%. Two hundred fifty of these strips are acquired today at a cost of \$86,640.

What is the risk of loss of principal in year 18 if interest rates rise to 8.06% at the time of redemption ?

The answer is zero risk, since the bonds will be redeemed at \$1,000 per bond. The risk inherent in this scenario is the calculation of the amount which will be required in year 18. This calculation must anticipate a reasonable inflation rate, since tuition fees in 18 years will not be at today's prices. But if held to maturity, there is very little risk¹⁰ in receiving the *maturity value* of a U.S. Treasury bond. The *purchasing power* of the recovered principal, however, is another matter.

¹⁰ Short of a sovereign default.